Perturbed Coulombic potentials in Dirac and Klein-Gordon equations

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Abstract

A relativistic extension of our pseudo-shifted ℓ–expansion technique is presented to solve for the eigenvalues of Dirac and Klein-Gordon equations. Once more we show the numerical usefulness of its results via comparison with available numerical integration data.

I. INTRODUCTION

Dirac and Klein-Gordon (KG) equation are not exactly soluble for most of Lorentz vector (coupled as the 0-component of the 4-vector) and/or Lorentz scalar (added to the mass term) potentials[1-14]. One, therefore, has to resort to some approximation schemes [1-6]. Yet, in between non-numerical and purely numerical non-relativistic (Schrödinger) and relativistic (KG and Dirac) wave equations there exists a broad *gray zone* of potentials tractable via various systematic semi-numerical (or semi-analytical) power-series expansions.

In numerous methodical predecessors of a subset of papers, Mustafa and co-workers [7] have sought a possibility in the power-law asymptotic expansions using some *small* parameter to solve for Schrödinger equation. It has been noted that the presence of the central spike $\ell_d\,(\ell_d+1)\,/\,r^2$ (where $\ell_d=\ell+(d-3)/2$ and the dimensions $d\geq 2$) in the radial Schrödinger equation, just copies the effect of the centrifugal and/or centripetal force and immediately inspires the use of *small shifted inverse angular momentum quantum number*. Their PSLET (pseudo-perturbative shifted-ℓ expansion technique) has provided persuasive numerical verifications by immediate comparison of its results with available *brute force* numerical data [7]. PSLET simply consists of using $1/\bar{l}$ as a perturbative expansion parameter, where $\bar{l}=\ell-\beta_o$, ℓ is a quantum number, and β_o is a suitable shift introduced to avoid the trivial case $\ell=0$ [7].pagebreak

In this paper, we extend PSLET recipe to solve for Dirac and KG equations with Lorentz scalar and/or Lorentz vector radially symmetric potentials (in section 2). In section 3 we apply this relativistic recipe to some exactly solvable potentials (e.g., $V(r)=S(r)=-A/r$, $V(r)=-A_1/r$ and $S(r)=0$, $V(r)=0$ and $S(r)=-A_2/r$, and the Dirac oscillator) and non-exactly solvable (by PSLET) potentials (e.g., the pure scalar linear, the *funnel-shapped*, and the power-law potentials) to study the usefulness of its numerical results. We conclude in section 4.

II. PSLET RECIPE FOR DIRAC AND KG EQUATIONS

The Dirac equation with the Lorentz scalar (added to the mass term) and Lorentz vector (coupled as the 0-component of the 4-vector potential) potentials reads (in $\hbar=c=1$ units)

$$\{\vec{\alpha}.\vec{p}+\beta\,[m+S(r)]\}\;\Psi(\vec{r})=\{E-V(r)\}\;\Psi(\vec{r}). \tag{1}$$

Which decouples into

$$\xi_1(r)\,G(r)+\frac{dF(r)}{dr}-\frac{\kappa}{r}F(r)=0, \tag{2}$$

$$\xi_2(r)\,F(r) - \frac{dG(r)}{dr} - \frac{\kappa}{r}G(r) = 0. \tag{3}$$

where $\kappa = -(\ell+1)$ for $j = \ell + 1/2$, $\kappa = \ell$ for $j = \ell - 1/2$, and

$$\xi_1(r) = E - V(r) - [m + S(r)]\,,$$

$$\xi_2(r) = E + m - y(r); \;\; y(r) = V(r) - S(r).$$

E is the relativistic energy, and $G(r)$ and $F(r)$ are the large and small radial components of the Dirac spinor, respectively. In terms of the large component $G(r)$, equation (2) reads

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{1}{\xi_2(r)}\,(\,y^{'}(r)\,[\frac{d}{dr} + \frac{\kappa}{r}]\,) + \xi_1(r)\,\xi_2(r)\right] G(r) = 0, \tag{4}$$

where the prime denotes d/dr. It can be shown that with the ansatz

$$G(r) = \Phi(r)\; exp(-p(r)\,/\,2)\;; \qquad p^{'}(r) = y^{'}(r)\,/\,\xi_2(r), \tag{5}$$

equation (4) reads

$$\left\{-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + U(r) - \xi_1(r)\,\xi_2(r)\right\}\Phi(r) = 0, \tag{6}$$

where

$$U(r) = \frac{y^{''}(r)}{2\,\xi_2(r)} - \frac{\kappa}{r}\frac{y^{'}(r)}{\xi_2(r)} + \frac{3}{4}\left(\frac{y^{'}(r)}{\xi_2(r)}\right)^2. \tag{7}$$

Obviously, equation (6) reduces to KG-equation with $\kappa\,(\kappa+1) = \ell\,(\ell+1)$, for any κ, if $U(r)$ is set zero. It is therefore convenient to introduce a parameter $\lambda = 0,\, 1$ in $U(r)$ so that $\lambda = 0$ and $\lambda = 1$ correspond to KG and Dirac equations respectively. Also, we shall be interested in the problems where the rest energy mc^2 is large compared to the binding energy $E_{bind.} = E - mc^2$. This would manifest the approximation

$$\frac{1}{\xi_2(r)} = \frac{1}{E_{bind.} + 2m - y(r)} \simeq \frac{1}{2m} - O(1/m^2), \tag{8}$$

which in turn implies

$$U(r) = \frac{\lambda}{4\,m}\left[y^{''}(r) - \frac{2\,\kappa\,y^{'}(r)}{r} + \frac{3\,y^{'}(r)^2}{4\,m}\right]. \tag{9}$$

For Coulombic-like potentials (i.e., a Lorentz-vector $V(r) = -A_1/r$ and a Lorentz-scalar $S(r) = -A_2/r$ potentials) one may re-scale the potentials and use the substitutions

$$V_r(r) = V(r)^2 - \frac{A_1^2}{r^2}, \tag{10}$$

$$S_r(r) = S(r)^2 - \frac{A_2^2}{r^2}, \tag{11}$$

to recast equation (6) as

$$\left[-\frac{d^2}{dr^2} + \frac{[\,\bar{l}^2 + \bar{l}\,(2\,\beta_o + 1) + \beta_o\,(\beta_o + 1)]}{r^2} + \Gamma(r) + 2\,E\,V(r)\right]\,\Phi(r) = E^2\Phi(r). \tag{12}$$

where

$$\Gamma(r) = -V_r(r) + S_r(r) + 2\,m\,S(r) + m^2 + U(r), \tag{13}$$

$$\bar{l} = \ell' - \beta_o; \quad \ell' = -\frac{1}{2} + \sqrt{(\ell + 1/2)^2 - A_1^2 + A_2^2}, \tag{14}$$

and β_o is a suitable shift to be determined below. Next, we shift the origin of the coordinate system $x = \bar{l}^{1/2}(r - r_o)/r_o$, where r_o is currently an arbitrary point to be determined through the minimization of the leading energy term below. It is therefore convenient to expand about $x = 0$ (i.e., $r = r_o$) and use the following expansions

$$r^{-2} = \sum_{n=0}^{\infty} \frac{a_n}{r_o^2}\,x^n\,\bar{l}^{-n/2};\ a_n = (-1)^n\ (n+1), \tag{15}$$

$$\Gamma(x(r)) = \frac{\bar{l}^2}{Q}\,\sum_{n=0}^{\infty}\,b_n\,x^n\,\bar{l}^{-n/2};\ b_n = \frac{d^n\,\Gamma(r_o)}{dr_o^n}\,\frac{r_o^n}{n!} \tag{16}$$

$$V(x(r)) = \frac{\bar{l}}{\sqrt{Q}}\,\sum_{n=0}^{\infty}\,c_n\,x^n\,\bar{l}^{-n/2};\ c_n = \frac{d^n\,V(r_o)}{dr_o^n}\,\frac{r_o^n}{n!} \tag{17}$$

$$E = \frac{1}{\sqrt{Q}}\,\sum_{n=-1}^{\infty}\,E^n\,\bar{l}^{-n}, \tag{18}$$

where Q is set equal to $\bar{l}^2$ at the end of the calculations. With the above expressions into (12), one may collect all x-dependent terms of order $\bar{l}$ to imply the leading-order approximation for the energies

$$E^{(-1)} = V(r_o) \pm \sqrt{V(r_o)^2 + \Gamma(r_o) + \frac{Q}{r_o^2}}. \tag{19}$$

Which upon minimization, i.e., $dE^{(-1)}/dr_o = 0$ and $d^2E^{(-1)}/dr_o^2 > 0$, yields

$$2\,Q = h\,(r_o) + \sqrt{h\,(r_o)^2 - g\,(r_o)} \tag{20}$$

where

$$h(r_o) = r_o^3\left[2\,V(r_o)\,V^{'}(r_o) + \Gamma^{'}(r_o) + r_o V^{'}(r_o)^2\right], \tag{21}$$

$$g(r_o) = r_o^6 \left[\Gamma^{'}(r_o)^2 + 4V(r_o)V^{'}(r_o)\Gamma^{'}(r_o) - 4\Gamma(r_o)V^{'}(r_o)^2\right] \tag{22}$$

and primes denote derivatives with respect to r_o. This implies that $x\bar{l}^{-1}$-coefficients vanish, i.e.;

$$Q\, a_1 + r_o^2\, b_1 + 2\, r_o^2\, E^{(-1)}\, c_1 = 0. \tag{23}$$

Equation (12) therefore reduces to

$$\begin{aligned}
&\left[-\frac{d^2}{dx^2} + \sum_{n=2}^{\infty} T_n\, x^n \bar{l}^{-(n-2)/2} + (2\,\beta_o + 1)\sum_{n=0}^{\infty} a_n\, x^n\, \bar{l}^{-n/2}\right. \\
&+\beta_o\,(\beta_o+1)\sum_{n=0}^{\infty} a_n\, x^n\, \bar{l}^{-(n+2)/2} \\
&\left.+\frac{2\, r_o^2}{Q}\sum_{n=0}^{\infty}\sum_{p=0}^{n+1} E^{(n-p)}\left(c_{2p}\, x^{2p}\, \bar{l}^{-n} + c_{2p+1}\, x^{2p+1}\bar{l}^{-(n+1/2)}\right)\right]\, \Phi_{k,\ell}(x) \\
&= \left[\frac{r_o^2}{Q}\sum_{n=-1}^{\infty}\sum_{p=-1}^{n+1} E^{(n-p)}\, E^{(p)}\, \bar{l}^{-(n+1)}\right]\, \Phi_{k,\ell}(x)
\end{aligned} \tag{24}$$

where

$$T_n = a_n + \frac{r_o^2}{Q}\, b_n. \tag{25}$$

One may now compare equation (24) with the Schrödinger equation for the one-dimensional anharmonic oscillator

$$\left[-\frac{d^2}{dy^2} + \frac{1}{4}\,\omega^2 y^2 + \varepsilon_o + B(y)\right]\, Y_k(y) = \mu_k\, Y_k(y) \tag{26}$$

where ε_o is constant, $B(y)$ is a perturbation like term and

$$\mu_k = \varepsilon_o + (k+1/2)\omega + \sum_{n=1}^{\infty} \mu^{(n)}\, \bar{l}^{-n}, \tag{27}$$

with $k = 0, 1, 2, \cdots$ and

$$\omega = \sqrt{12 + \frac{2r_o^4}{Q}\,\Gamma''(r_o) + \frac{4\, r_o^4}{Q}\, E^{(-1)}\, V''(r_o)}. \tag{28}$$

In a straightforward manner, one can show that

$$E^{(0)} = \frac{Q}{2r_o^2\,[E^{(-1)} - c_0]}\,[(2\,\beta_o + 1) + (k+1/2)\,\omega]\,, \tag{29}$$

and choose β_o so that $E^{(0)} = 0$ to obtain

$$\beta_o = -\frac{1}{2}\left[1 + (k + 1/2)\,\omega\right]. \tag{30}$$

equation (24) then becomes

$$\left[-\frac{d^2}{dx^2} + \sum_{n=0}^{\infty}\left(\upsilon^{(n)}(x)\,\bar{l}^{-n/2} + J^{(n)}(x)\,\bar{l}^{-n} + K^{(n)}(x)\,\bar{l}^{-(n+1/2)} + \epsilon^{(n)}\,\bar{l}^{-(n+1)}\right)\right]\Phi_{k,\ell}(x) = 0, \tag{31}$$

reak where

$$\upsilon^{(0)}(x) = T_2\,x^2 + (2\,\beta_o + 1)\;a_0, \tag{32}$$

$$\upsilon^{(1)}(x) = T_3\,x^3 + (2\,\beta_o + 1)\;a_1 x, \tag{33}$$

$$\upsilon^{(n)}(x) = T_{n+2}\,x^{n+2} + (2\,\beta_o + 1)\;a_n\,x^n + \beta_o\,(\beta_o + 1)\,a_{n-2}\,x^{n-2};\;\; n \geq 2, \tag{34}$$

$$J^{(n)}(x) = \left(\frac{2\,r_o^2}{Q}\right)\sum_{p=0}^{n+1} E^{(n-p)}\,c_{2p}\,x^{2p}, \tag{35}$$

$$K^{(n)}(x) = \left(\frac{2\,r_o^2}{Q}\right)\sum_{p=0}^{n+1} E^{(n-p)}\,c_{2p+1}\,x^{2p+1} \tag{36}$$

$$\epsilon^{(n)} = \left(\frac{r_o^2}{Q}\right)\sum_{p=-1}^{n+1} E^{(n-p)}\,E^{(p)}. \tag{37}$$

Now we may closely follow PSLET recipe for the k-nodal wavefunction and define

$$\Phi_{k,\ell}(x) = F_{k,\,\ell}(x)\;\exp(U_{k,\,\ell}(x)) \tag{38}$$

where

$$F_{k,\,\ell}(x) = x^k + \sum_{n=0}^{\infty}\sum_{p=0}^{k-1} A_{p,k}^{(n)}\,x^p\,\bar{l}^{-n/2}, \tag{39}$$

$$U'_{k,\,\ell}(x) = \sum_{n=0}^{\infty}\left(U_{k,\,\ell}^{(n)}(x)\;\bar{l}^{-n/2} + G_{k,\,\ell}^{(n)}(x)\;\bar{l}^{-(n+1)/2}\right), \tag{40}$$

with

$$U_{k,\ell}^{(n)}(x) = \sum_{p=0}^{n+1} D_{p,n,k}\, x^{2p-1}; \;\; D_{0,n,k} = 0, \tag{41}$$

$$G_{k,\ell}^{(n)}(x) = \sum_{p=0}^{n+1} C_{p,n,k}\, x^{2p}. \tag{42}$$

Equation (31) then reads

$$\begin{aligned} & F_{k,\ell}(x) \sum_{n=0}^{\infty} [\upsilon^{(n)}(x)\, \bar{l}^{-n/2} + J^{(n)}(x)\, \bar{l}^{-n} + K^{(n)}(x)\, \bar{l}^{-(n+1/2)} - \epsilon^{(n)}\, \bar{l}^{-(n+1)}] \\ & -F_{k,\ell}(x)\, [U_{k,\ell}^{''}(x) + U_{k,\ell}^{'}(x) U_{k,\ell}^{'}(x)] - 2\, F_{k,\ell}^{\prime}(x)\, U_{k,\ell}^{'}(x) - F_{k,\ell}^{''}(x) = 0 \end{aligned} \tag{43}$$

where primes denote derivatives with respect to x. One may also eliminate $\bar{l}$-dependance from equation (43) to obtain four exactly solvable recursive relations (see Appendix for details). Once r_o is determined, through equation (20), one may then calculate the energy eigenvalues and eigenfunctions from the knowledge of $C_{p,n,k}$, $D_{p,n,k}$ and $A_{p,k}^{(n)}$ in a hierarchical manner.

III. ILLUSTRATIVE EXAMPLES

In this section we illustrate the applicability of the above relativistic PSLET recipe through some examples covering Dirac and KG equations.

A. An equally mixed Coulomb potentials

For an equally mixed Coulombic potentials, i.e. $V(r) = S(r) = -A/r$, $U(r)$ in (9) vanishes. Consequently $\Gamma(r) = -2mA/r + m^2$, $Q = -A^2 + 2mAr$, $\omega = 2$, $\beta_o = -(k+1)$, $r_o = \left[(\ell' + k + 1)^2 + A^2\right] / (2mA)$, and the leading-order approximation reads

$$E^{(-1)} = m \left[1 - \frac{2\, A^2}{(k + \ell + 1)^2 + A^2}\right],$$

which is the well known exact result for the generalized Dirac- and KG-Coulomb problems, where higher-order corrections vanish identically.

B. Vector Coulomb or scalar Coulomb potential

For $V(r) = -A_1/r$ and $S(r) = 0$ or $V(r) = 0$ and $S(r) = -A_2/r$ in KG equation one would obtain the well known exact results

$$E^{(-1)} = m \left[1 + \frac{A_1^2}{n_1^2}\right]^{-1/2} ; \;\; n_1 = k + \frac{1}{2} + \sqrt{(\ell + 1/2)^2 - A_1^2}$$

or

$$E^{(-1)} = \pm\, m \left[1 - \frac{A_2^2}{n_2^2}\right]^{1/2} ; \;\; n_2 = k + \frac{1}{2} + \sqrt{(\ell + 1/2)^2 + A_2^2}$$

respectively. Again higher-order corrections vanish identically.

C. Dirac oscillator

Following the work of Romero et al [13], the Dirac oscillator [14] eigenvalue problem (see equation (30) in [13]) reduces to

$$\left[-\frac{d^2}{dr^2} + \frac{\Lambda(\Lambda + \epsilon\,\beta)}{r^2} + m^2B^2r^2 + m^2 + mB\left(\epsilon\left[2j+1\right] - \beta\right)\right]\Phi(r) = E^2\Phi(r),$$

where $\Lambda = j + 1/2$, B is the oscillator frequency, and $\epsilon = \pm 1$. In this case our $\Gamma(r) = m^2B^2r^2 + m^2 + mB\left(\epsilon\left[2j+1\right] - \beta\right)$, $\omega = 4$, $r_o^2 = \bar{l}\,/mB$, and our leading term reads

$$E^{(-1)} = \pm\, \left[2mB\left(2k + \ell + 3/2\right) + m^2 + mB\left(\epsilon\left[2j+1\right] - \beta\right)\right]^{1/2}$$

with heigher-order terms identical zeros. Thus, if we take $N = 2k + \ell$ (the harmonic oscillator principle quantum number) we come out with the exact Dirac oscillator's closed form solution (see equation (35) in [13])

$$E^2 - m^2 = \left[2mB\left(N + 3/2\right) + mB\left(\epsilon\left[2j+1\right] - \beta\right)\right].$$

D. Pure scalar linear potential

A pure scalar linear potential, i.e., $S(r) = Ar$ and $V(r) = 0$, is precisely a quark confining potential. It has been used by Gunion and Li [5] in Dirac equation to find, numerically, part of Dirac J/Ψ mass spectra.

Obviously, for this potential equation (20) has to be solved numerically. Then one can proceed to obtain the mass spectra for $A = 0.137\,GeV^2$, $m = 1.12\,GeV$, $\kappa = -\left(\ell + 1\right)$, and $\kappa = \ell$ through the prescription $M = 2E$.

In tables 1 and 2 we report our results for J/Ψ mass (in GeV) for $\kappa = -\left(\ell + 1\right)$ and $\kappa = \ell$, respectively. To show the trends of convergence of our results, we report them as $M(N) = 2E(N)$, with N denoting the number of corrections added to the leading-order approximation $E^{(-1)}$. Our results are also compared with the numerically predicted ones of Gunion and Li [5]. Evidently, the accuracy and trend of convergence are satisfactory.

E. *Funnel-shaped* potential

The *funnel-shaped* potential is widely used in quarkonium physics. It has both vector and components, $V(r)$ and $S(r)$, respectively.

In Dirac equation, Stepanov and Tutik [4] have used numerical integrations and $\hbar$-expansion formalism without the traditional conversion of Dirac equation into a Schrődinger-like form (unlike what we have already done in section II above). They have obtained the Charmonium masses for $V(r) = -2\alpha/3r$ and $S(r) = br/2$, where $m = 1.358\,GeV$, $\alpha = 0.39$, and $b = 0.21055\,GeV^2$.

In table 3 we show our results for the Charmonium masses for $\kappa = \ell$ and compare them with those of numerical integration and $\hbar$-expansions of Stepanov and Tutik [4]. They are in good agreement and the trend of convergence of our results is also satisfactory. However, in table 4 we report the Charmonium masses, for $\kappa = -(\ell + 1)$. Therein, we only list our results where the mass series and Padé approximants stabilize.

In KG equation Kobylinsky, Stepanov, and Tutik [3] have used $V(r) = -a/r$ and $S(r) = b\,r$ with $m = 1.370\,GeV$, $b = 0.10429\,GeV^2$ and $a = 0.26$ to obtain the energies for this *funnel-shaped* potential. They have also used $\hbar$-expansions and numerical integrations. In table 5 we list our results and compare them with those of $\hbar$-expansions, $E_\hbar$, and numerical integrations, E_{num}, reported in [3]. They are in excellent agreement.

F. Power-law potential

An equally mixed scalar and vector power-law potential

$$V(r) = S(r) = Ar^\nu + B_o$$

where $\nu = 0.1$ (the Martin potential [6]) and $A > 0$ is found most successful in describing the entire light and heavy meson spectra in the Dirac equation (cf. e.g., Martin 1980, 1981, and Jena and Tripati in [6]). Once such potential setting is used in Eq.(6) along with the substitution $q = r/\varrho$, with

$$\varrho = \left[2\,(E + m)\,A\right]^{-1/(\nu+2)},$$

one gets a simple Schrödinger-type form

$$\left[-\frac{d^2}{dq^2} + \frac{\ell\,(\ell+1)}{q^2} + q^\nu\right]\,\Omega\,(q) = \check{E}\,\Omega\,(q)\,, \tag{44}$$

where

$$\check{E} = (E - m - 2B_o)\left[(E + m)\,(2A)^{-2/\nu}\right]^{\nu/(\nu+2)}. \tag{45}$$

Therefore, one better solve (44) for $\check{E}\,(N)$ and then to find the Dirac quark binding energies E from (45). In table 6 we compare PSLET results with those obtained numerically, E_{num}, by Jena and Tripati [6]. The results from the shifted $1/N$ - expansion technique by Roy and Roychoudhury [6], $E_{1/N}$, are also listed.

IV. CONCLUDING REMARKS

In this work we presented a straightforward extension of our PSLET recipe [7] to solve for the eigenvalues of Dirac and KG equations, with Lorentz vector and/or Lorentz scalar potentials. We have, again, documented (through tables 1-5) the usefulness of this recipe by immediate comparisons with available numerical integration data.

Nevertheless, one should notice that our results from KG equation are only partially better , compared with those from $\hbar$-expansions and numerical integrations, than our results from Dirac equation. The reason is obviously, and by large, manifested by our approximation in equation (8). For Klein-Gordon equation $\lambda = 0$ in (9) while for Dirac equation $\lambda = 1$.

In the process, moreover, there still remain some issues of delicate nature. Namely, one can not obtain (using our PSLET above) the exact eigenvalues for $V(r) = -A_1/r$ and $S(r) = 0$, $V(r) = 0$ and $S(r) = -A_2/r$, or even for $V(r) = -A_1/r$ and $S(r) = -A_2/r$ in Dirac equation.

The remedy seems to be feasible in a sort of combination between the current relativistic PSLET and a similarity transformation (cf., e.g. ref.[8] and related references therein). Preliminary results show that if $S(r) \longrightarrow -A_2/r + S_o(r)$ and $V(r) \longrightarrow -A_1/r + V_o(r)$ in (1) such that $S_o(r) \longrightarrow 0$ and $V_o(r) \longrightarrow 0$ as $r \longrightarrow 0$, then a similarity transformation could accompany our relativistic PSLET to obtain exact results for the generalized Dirac-Coulombic problem and better results for potentials of the type $S(r) \longrightarrow -A_2/r + S_o(r)$ and $V(r) \longrightarrow -A_1/r + V_o(r)$. That is, one may carefully follow Mustafa's work [8] to obtain

$$\left[-\frac{d^2}{dr^2} + \frac{(\gamma^2 + s\,\gamma)}{r^2} + U(r) + m^2 + \frac{2}{r}\,(A_2\,m(r) + E(r)\,A_1)\right]\,\Phi(r) = E^2\Phi(r). \tag{46}$$

where $\gamma = \sqrt{\kappa^2 - A_1^2 + A_2^2}$, $s = \pm 1$, $E(r) = E - V_o(r)$, $m(r) = m + S_o(r)$, and $U(r) \longrightarrow 0$ as $V_o(r) \longrightarrow 0$ and $S_o(r) \longrightarrow 0$ (hence, $m(r) \longrightarrow m$ and $E(r) \longrightarrow E$). Therefore, one would replace our ℓ', in (12), by $\tilde{\ell} = -1/2 + \gamma + s\,/\,2$ and obtain (following PSLET recipe above) for (48), $\Gamma(r_o) = m^2 - 2mA_2r_o$, $V(r) = -A_1/r$, $\omega = 2$, $\beta_o = -\,(k+1)$, and $\bar{l} = \tilde{\ell} - \beta_o = (k + 1/2 + s\,/\,2 + \gamma)$. In turn, equation (23) yields

$$4Q\left[Q - 2mA_2r_o + A_1^2\right] + 4m^2r_o^2\left[A_2^2 - A_1^2\right] = 0 \tag{47}$$

to solve for r_o. This would lead to

$$E^{(-1)} = m\left[1 + \frac{A_1^2}{Q}\right]^{-1/2} \;;\;\; Q = \left(k + 1/2 + s\,/\,2 + \sqrt{\kappa^2 - A_1^2}\right)^2 \tag{48}$$

and

$$E^{(-1)} = \pm\, m\left[1 - \frac{A_2^2}{Q}\right]^{1/2} \;;\;\; Q = \left(k + 1/2 + s\,/\,2 + \sqrt{\kappa^2 + A_2^2}\right)^2 \tag{49}$$

for $A_2 = 0$, $A_1 \neq 0$ and $A_1 = 0$, $A_2 \neq 0$, respectively. It should be noted that these are the well known exact results (cf., e.g.; ref [8]) with the heigher-order terms vanish identically.

Before we conclude it should be noted that if our results are to be generalized to d-dimensions we may incorporate interdimensional degeneracies associated with the isomorphism between the angular momentum ℓ and dimensionality d (cf., e.g., Mustafa and Odeh

(2000) [7]). This would replace our κ by $\kappa_d = s(2j + d - 2)/2$, where $\ell_d = \ell + (d - 3)/2$. In this way we reproduce Stepanov and Tutik's [4] and Dong's [9] results in d-dimensions.

Finally, the above has been a very limited review and a number of other useful and novel approaches such as those of Franklin [10, and references therein], Njock et al [11, and references therein], $\cdots$etc., have not been touched on.

Appendix A: Further algebraic simplifications for PSLET relativistic recipe

Eliminating $\bar{l}$-dependence, equation (42) can be simplified into four recursive relations to read

$$k(k-1)\,x^{k-2} + T^{(0)}_{k,\ell}(x) - N^{(0)}_{k,\ell}(x) = 0 \tag{50}$$

$$T^{(1)}_{k,\ell}(x) + S^{(0)}_{k,\ell}(x) - O^{(0)}_{k,\ell}(x) = 0 \tag{51}$$

and for $n \geq 0$

$$T^{(2n+2)}_{k,\ell}(x) - N^{(n+1)}_{k,\ell}(x) + S^{(2n+1)}_{k,\ell}(x) + M^{(2n)}_{k,\ell}(x) + \Lambda^{(n)}_{k,\ell}(x) = 0 \tag{52}$$

$$T^{(2n+3)}_{k,\ell}(x) + S^{(2n+2)}_{k,\ell}(x) - O^{(n+1)}_{k,\ell}(x) + M^{(2n+1)}_{k,\ell}(x) + \zeta^{(n)}_{k,\ell}(x) = 0 \tag{53}$$

where

$$\begin{aligned} T^{(n)}_{k,\ell}(x) = {} & L^{(n)''}_{k,\ell}(x) + 2\,k\,x^{k-1} U^{(n)}_{k,\ell}(x) \\ & + \sum_{p=0}^{n} 2 L^{(p)'}_{k,\ell}(x)\, U^{(n-p)}_{k,\ell}(x) + x^k \left[U^{(n)'}_{k,\ell}(x) + R^{(n)}_{k,\ell}(x) - \upsilon^{(n)}(x) \right] \\ & + \sum_{p=0}^{n} L^{(p)}_{k,\ell}(x) \left(U^{(n-p)'}_{k,\ell}(x) + R^{(n-p)}_{k,\ell}(x) - \upsilon^{(n-p)}(x) \right) \end{aligned} \tag{54}$$

$$\begin{aligned} S^{(n)}_{k,\ell}(x) = {} & 2\,k\,x^{k-1} G^{(n)}_{k,\ell}(x) + \sum_{p=0}^{n} 2 L^{(p)'}_{k,\ell}(x)\, G^{(n-p)}_{k,\ell}(x) \\ & + x^k \left[G^{(n)'}_{k,\ell}(x) + Q^{(n)}_{k,\ell}(x) \right] + \sum_{p=0}^{n} L^{(p)}_{k,\ell}(x) \left(G^{(n-p)'}_{k,\ell}(x) + Q^{(n-p)}_{k,\ell}(x) \right) \end{aligned} \tag{55}$$

$$M^{(n)}_{k,\ell}(x) = x^k\, P^{(n)}_{k,\ell}(x) + \sum_{p=0}^{n} L^{(p)}_{k,\ell}(x)\; P^{(n-p)}_{k,\ell}(x) \tag{56}$$

$$N^{(n)}_{k,\ell}(x) = x^k\, J^{(n)}(x) + \sum_{p=0}^{n} L^{(p)}_{k,\ell}(x)\; J^{(n-p)}(x) \tag{57}$$

$$O^{(n)}_{k,\ell}(x) = x^k\, K^{(n)}(x) + \sum_{p=0}^{n} L^{(p)}_{k,\ell}(x)\; K^{(n-p)}(x) \tag{58}$$

$$\Lambda^{(n)}_{k,\ell}(x) = x^k\, \epsilon^{(n)} + \sum_{p=0}^{n} L^{(2p)}_{k,\ell}(x)\, \epsilon^{(n-p)} \tag{59}$$

$$\zeta_{k,\ell}^{(n)}(x) = \sum_{p=0}^{n} L_{k,\ell}^{(2p+1)}(x)\, \epsilon^{(n-p)} \tag{60}$$

$$R_{k,\ell}^{(n)}(x) = \sum_{p=0}^{n} U_{k,\,\ell}^{(p)}(x) U_{k,\,\ell}^{(n-p)}(x) \tag{61}$$

$$P_{k,\ell}^{(n)}(x) = \sum_{p=0}^{n} G_{k,\,\ell}^{(p)}(x) G_{k,\,\ell}^{(n-p)}(x) \tag{62}$$

$$Q_{k,\ell}^{(n)}(x) = \sum_{p=0}^{n} 2U_{k,\,\ell}^{(p)}(x) G_{k,\,\ell}^{(n-p)}(x) \tag{63}$$

$$L_{k,\ell}^{(n)}(x) = \sum_{p=0}^{k-1} A_{k,\,p}^{(n)}(x)\, x^{p} \tag{64}$$

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Table1: PSLET results for part of Dirac J/Ψ spectra (in GeV) for $S(r) = Ar$, $A = 0137GeV^2$, $V(r) = 0$, $\kappa = -(\ell + 1)$, and $m = 1.12GeV$. Where $M(N) = 2E(N)$ with N denoting the number of corrections added to the leading - order term $E^{(-1)}$ and M_{num} are the numerical values reported by Gunion and Li [5].

k	$M(N)$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$
0	$M(1)$	3.0919	3.43078	3.711960	3.9581219
	$M(2)$	3.0961	3.43252	3.712914	3.9587277
	$M(3)$	3.0963	3.43259	3.712947	3.9587465
	$M(4)$	3.0961	4.43256	3.712940	3.9587436
	$M(5)$	3.0961	3.43256	3.712939	3.9587434
	⋮	⋮	⋮	⋮	⋮
	$M(14)$	3.0961	3.43256	3.712939	3.9587434
	M_{num}	3.103	3.442	3.725	3.973
2	$M(1)$	4.131	4.3395	4.5325	4.71334
	$M(2)$	4.142	4.3468	4.5378	4.71739
	$M(3)$	4.148	4.3502	4.5401	4.71905
	$M(4)$	4.150	4.3515	4.5408	4.71950
	$M(5)$	4.151	4.3519	4.5410	4.71961
	$M(6)$	4.152	4.3520	4.5411	4.71965
	$M(7)$	4.152	4.3521	4.5411	4.71966
	⋮	⋮	⋮	⋮	⋮
	$M(14)$	4.152	4.3521	4.5411	4.71966
	M_{num}	4.158	4.36	4.551	4.732

Table 2: Same as table 1 for $\kappa = \ell$.

k	$M(N)$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
0	$M(1)$	3.47090	3.760125	4.0111817	4.2364739
	$M(2)$	3.47183	3.760677	4.0115488	4.2367365
	$M(3)$	3.47188	3.760700	4.0115615	4.2367444
	$M(4)$	3.47186	3.760696	4.0115597	4.2367435
	$M(5)$	3.47186	3.760695	4.0115595	4.2367435
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$M(14)$	3.47186	3.760695	4.0115595	4.2367435
	M_{num}	3.47	3.757	4.006	4.23
1	$M(1)$	3.9570	4.19083	4.40310	4.599080
	$M(2)$	3.9624	4.19451	4.40578	4.601126
	$M(3)$	3.9640	4.19537	4.40631	4.601482
	$M(4)$	3.9644	4.19557	4.40642	4.601542
	$M(5)$	3.9646	4.19561	4.40644	4.601552
	$M(6)$	3.9646	4.19563	4.40644	4.601554
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	$M(14)$	3.9646	4.19563	4.40644	4.601554
	M_{num}	3.965	4.194	4.403	4.597

Table 3: PSLET Charmonium masses $M(N) = 2E(N)$ for the *funnel-shaped* potential, $V(r) = -2\alpha/3r$ and $S(r) = br/2$, with $m = 1.358\,GeV$, $b = 0.21055\,GeV^2$, $\alpha = 0.39$ and $\kappa = \ell$. The quantum numbers in parentheses are (k, ℓ). M_{num} is the numerical integration and $M_\hbar$ is the $\hbar$-expansion result (up to the tenth-order correction) reported by Stepanov and Tutik [4].

$M(N)$	$(0,1)$	$(0,2)$	$(1,1)$	$(1,3)$	$(2,1)$	$(2,3)$
$M(1)$	3.5071	3.8012	3.966	4.3862	4.333	4.6906
$M(2)$	3.5062	3.8007	3.963	4.3857	4.331	4.6908
$M(3)$	3.5056	3.8006	3.961	4.3853	4.329	4.6905
$M(4)$	3.5055	3.8005	3.959	4.3852	4.327	4.6901
$M(5)$	3.5055	3.8005	3.959	4.3851	4.326	4.6899
$M(6)$	3.5055	3.8005	3.958	4.3851	4.325	4.6898
$M(7)$	3.5055	3.8005	3.958	4.3850	4.325	4.6897
$M(8)$	3.5055	3.8005	3.958	4.3850	4.324	4.6897
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$M(14)$	3.5055	3.8005	3.958	4.3850	4.324	4.6897
$M_\hbar$	3.4998	3.7974	3.9501	4.3812	4.316	4.6858
M_{num}	3.4998	3.7974	3.9499	4.3812	4.315	4.6858

Table 4: Same as table 3 for $\kappa = -(\ell+1)$. Here we report the Charmonium masses where the mass series and Padé approximants $M_{i,j}$ stabilize.

k, ℓ	$M(N)$	$M[i,j]$	k, ℓ	$M(N)$	$M[i,j]$
0,0	M(6)=3.0333	M[4,4]=3.0333	1,0	M(7)=3.65	M[5,5]=3.6502
0,1	M(5)=3.4918	M[2,3]=3.4918	1,1	M(7)=3.946	M[4,4]=3.9462
0,2	M(4)=3.7787	M[2,3]=3.7787	1,2	M(7)=4.1690	M[4,4]=4.1690
0,3	M(4)=4.0129	M[2,3]=4.0129	2,0	M(9)=4.08	M[6,6]=4.0789
0,4	M(4)=4.2177	M[2,3]=4.2177	2,1	M(9)=4.314	M[4,5]=4.3139

Table 5: KG results for the *funnel-shaped* potential $S(r) = br$ and $V(r) = -a/r$, with $m = 1.370\,GeV$, $b = 0.10429\,GeV^2$, $a = 0.26$, $E_\hbar$ represents the results of Kobylinsky et al [3] via $\hbar$-expansion (up to the third-order correction), and E_{num} is the numerical integration value reported in [3].

$E(N)$	$k = 0,\ \ell = 0$	$k = 0,\ \ell = 1$	$k = 0,\ \ell = 2$
$E(1)$	1.541	1.76167	1.90420
$E(2)$	1.535	1.76064	1.90388
$E(3)$	1.534	1.76037	1.90380
$E(4)$	1.533	1.76033	1.90379
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$E(14)$	1.533	1.76033	1.90379
$E_\hbar$	1.536	1.7604	1.9038
E_{num}	1.533	1.760	1.904

Table 6: PSLET results for $\check{E}$ of equation (44) along with the numerically predicted ones by Jena and Tripati [6] and the $1/N$-expansion method $E_{1/N}$ by Roy and Roychoudhury [6]. N in $\check{E}(N)$ denotes the number of corrections added to the leading term where the series stabilizes.

k, ℓ	$\check{E}(N)$	E_{num}	$E_{1/N}$
0, 0	$\check{E}(2)$ =1.2358	1.2364	1.240
1, 0	$\check{E}(7)$ =1.3347	1.3347	1.340
2, 0	$\check{E}(4)$ =1.3922	1.3923	1.398
0, 1	$\check{E}(1)$ =1.3072	1.3071	1.309
1, 1	$\check{E}(4)$ =1.3731	1.3731	1.411
0, 2	$\check{E}(1)$ =1.3540	1.3544	1.358